Complex Numbers

$$z = a + bi = r\angle\theta$$

$$z^* = a - bi = r\angle -\theta$$

$$zz^* = a^2 + b^2 = r^2$$

TUTORIAL 4

1. The Complex Conjugate
2. Geometric Interpretation
3. Rationalization Processes
4. Illustrative Examples

G. David Boswell | Chronicles of BÖ§ZïK Inc.™

The 21st Century

Complex Conjugate, z*

Representation of ℂ-numbers

✳ The conjugate of a complex number makes it easy for us to do 2 very important algebraic operations:

- *Rationalization*
- *Division of Complex Numbers*

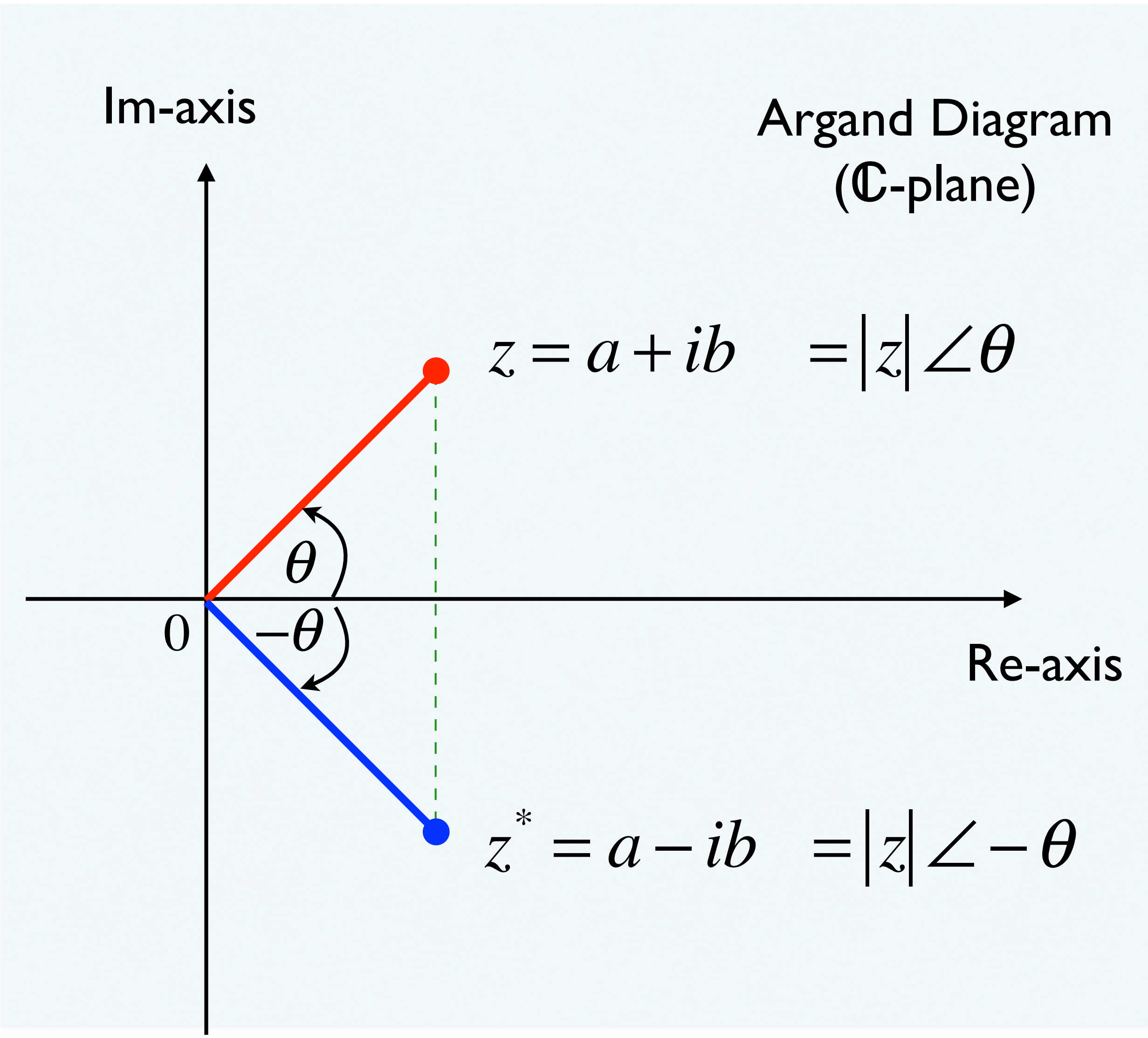
✳ The conjugate of a complex number is its reflection in the real axis.

✳ To form the conjugate of a complex number, we negate its imaginary part or its angle.

✳ The conjugate of z is written as z^* or $\bar{z}$.

Conjugate: Object & Image

Representation of ℂ-numbers



- ✳ The conjugate of a complex number is its reflection in the real axis.
- ✳ To form the conjugate of a complex number, we negate its imaginary part or its angle.

Complex Conjugate, z*

Representation of ℂ-numbers

Example 5 State the conjugate of the following complex numbers

$(a)\qquad z = 2 + 3i$

$(b)\qquad z = -5i$

$(c)\qquad z = 7.3\angle -2.4^c$

$(e)\qquad z = 9\angle 60^\circ$

$(f)\qquad z = 11\left(\cos 160^\circ + i\sin 160^\circ\right)$

$(g)\qquad z = 10e^{0.2i}$

Response 5 For each, we <u>negated</u> the imaginary part or the argument.

$(a)\qquad z^* = 2 - 3i$

$(b)\qquad z^* = 5i$

$(c)\qquad z^* = 7.3\angle 2.4^c$

$(e)\qquad z^* = 9\angle -60^\circ$

$(f)\qquad z^* = 11\left(\cos 160^\circ - i\sin 160^\circ\right)$

$(g)\qquad z^* = 10e^{-0.2i}$

Complex Conjugate, z*

Representation of ℂ-numbers

Question 6. State and read the conjugate, z^*, of the following complex numbers.

(a) $z = -2 - 4i$

(b) $z = 12 + 15i$

(c) $z = 3 - 8i$

(d) $z = 54\angle 90^\circ$

(e) $z = 11\angle 27.1^\circ$

(f) $z = \sqrt{5}\angle 4^c$

(g) $z = 2i\sin 6^\circ$

(h) $z = \cos 75^\circ - i\sin 75^\circ$

(i) $z = 2\left(\cos 15^\circ + i\sin 15^\circ\right)$

(j) $z = 5e^{4i}$

(k) $z = 25e^{\frac{\pi i}{8}}$

(l) $z = 100e^{i}$

No calculations necessary!

Rationalization Process
Use of Conjugate, z*

Example 7 Show that a *complex number multiplied by its conjugate* is a real number for the following cases:

$$(a) \qquad z = a + bi, \qquad a, b \in \mathbb{R}$$

$$(b) \qquad z = r\angle\theta, \qquad r \in \mathbb{R}$$

$$(c) \qquad z = 4 + 3i$$

$$(d) \qquad z = 8\angle 35^\circ$$

(In other words, show that $zz^* \in \mathbb{R}$)

Rationalization Process
Use of Conjugate, z*

Response 7(a).

A complex number times its conjugate equals the Sum of Squares!

Given $z = a + bi, \qquad a, b \in \mathbb{R}$

then $z^* = a - bi$

Hence
$$\begin{aligned} zz^* &= (a+bi)(a-bi) && \text{apply the distributive law} \\ &= a(a-bi) + bi(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 && \text{but } i^2 = -1 \\ &= a^2 + b^2 \\ &\equiv |z|^2 && \text{a real number!} \end{aligned}$$

Rationalization Process
Use of Conjugate, z*

Response 7(b).

A complex number times its conjugate equals the Sum of Squares!

Given $z = r\angle\theta, \qquad r \in \mathbb{R}$

then $z^* = r\angle -\theta$

Hence $zz^* = (r\angle\theta)(r\angle-\theta)$ multiply modulii & add angles

$= r^2\angle(\theta-\theta)$

$= r^2\angle 0$ (0° is on the real axis)

$= r^2$

$\equiv |z|^2$ a real number!

Rationalization Process

Use of Conjugate, z*

Response 7(c).

Given $z = 4 + 3i$

then $z^* = 4 - 3i$

Hence
$$zz^* = (4+3i)(4-3i)$$
no distributive law needed!
$$= 4^2 + 3^2$$
$$= 16 + 9$$
$$= 25$$
a real number!

Rationalization Process

Use of Conjugate, z*

Response 7(d).

Given $z = 8\angle 35^\circ$

then $z^* = 8\angle -35^\circ$

Hence $zz^* = \left(8\angle 35^\circ\right)\left(8\angle -35^\circ\right)$ mult. mod. & add angles

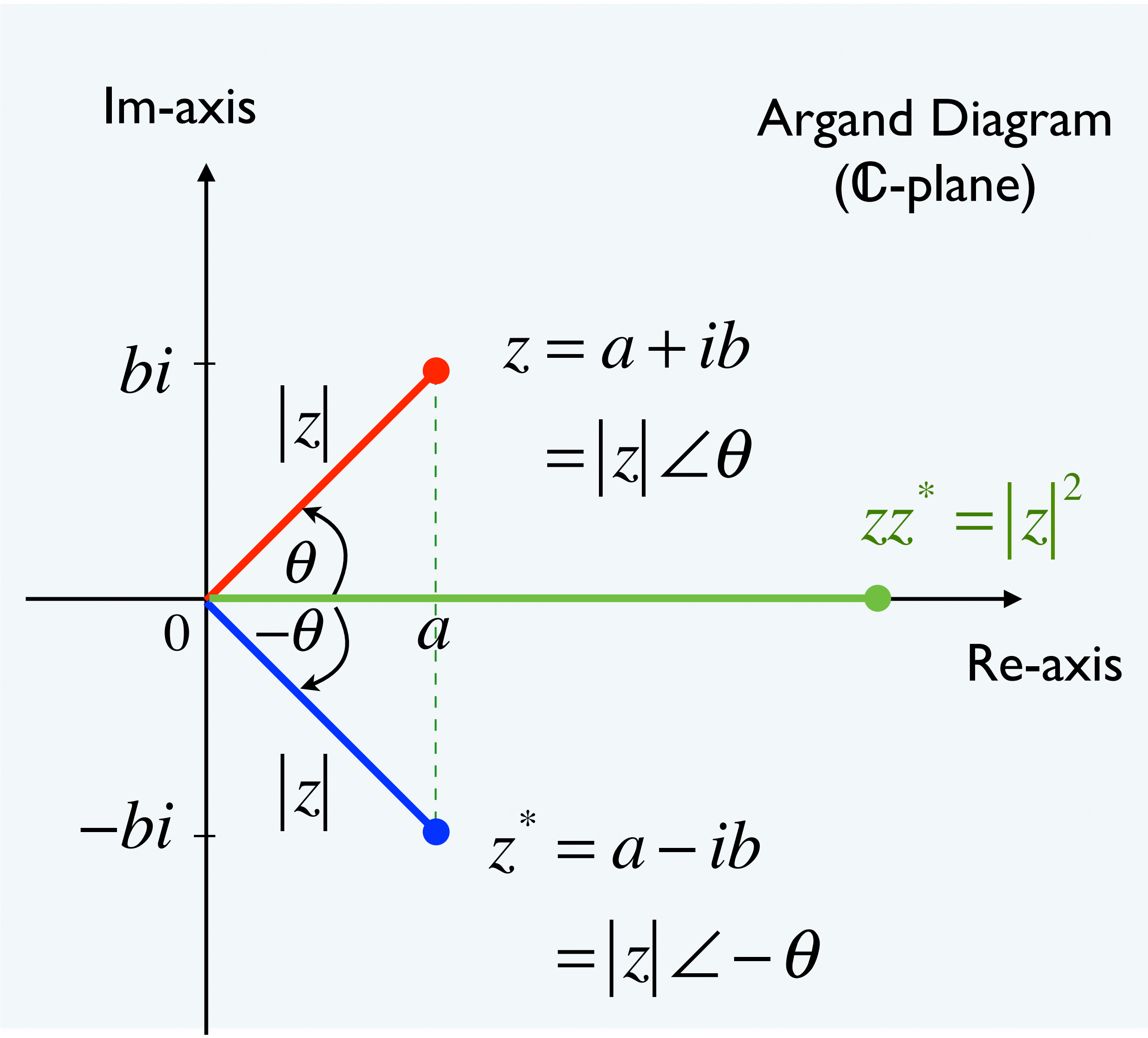
$= 8^2\angle 0^\circ$ (0° is on the real axis)

$= 64$ a real number!

A complex number times its conjugate equals the Sum of Squares!

Complex Rationalization

A Most Important Result



- ✳ A complex number multiplied by its conjugate equals the *Sum of Squares*!
- ✳ Hence,

$$\begin{aligned} zz^* &= (a+ib)(a-ib) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Complex Conjugate, z*

Representation of ℂ-numbers

Question 7.

For each of the following, compute the value of zz^*.

$(a)\qquad z = 1 + i$

$(b)\qquad z = i$

$(c)\qquad z = 3 + 4i$

$(d)\qquad z = 7\angle 15^\circ$

$(e)\qquad z = 2\left(\cos 60^\circ + i\sin 60^\circ\right)$

$(f)\qquad z = 9e^{2i}$

Division of Complex Numbers
Use of the Conjugate, z*

- In the *rectangular form*, division of complex numbers is achieved using *rationalization*

$$\frac{p+qi}{a+bi}=\left(\frac{p+qi}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right)=\frac{p}{a^2+b^2}+\frac{q}{a^2+b^2}i$$

- For the *polar form*, rationalization is not absolutely necessary! Simply divide the moduli and subtract the angles.

$$\frac{z_1}{z_2}=\frac{r_1\angle\theta_1}{r_2\angle\theta_2}=\frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

Division of Complex Numbers

Use of the Conjugate

Example 7 Simplify the following division processes using the *complex conjugate to rationalize the denominators*:

$(a) \quad z = \frac{2}{2+3i}$

$(b) \quad z = \frac{4+3i}{3-5i}$

$(c) \quad z = \frac{i}{1+i}$

$(d) \quad z = \frac{2}{i}$

Division of Complex Numbers

Use of the Conjugate

Response 7(a).

$$z = \frac{2}{2+3i}$$
$$= \left(\frac{2}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right)$$
$$= \frac{4-6i}{2^2+3^2}$$
$$= \frac{4}{13} - \frac{6}{13}i$$

- A complex number multiplied by its conjugate equals the *Sum of Squares*!
- Hence,

$$(a+ib)(a-ib)$$
$$= a^2 + b^2$$

Division of Complex Numbers

Use of the Conjugate

Response 7(b).

$$z = \frac{4+3i}{3-5i}$$

$$= \left(\frac{4+3i}{3-5i}\right)\left(\frac{3+5i}{3+5i}\right) \quad \text{apply distributive law to numerator}$$

$$= \frac{12+20i+9i+15i^2}{3^2+5^2} \quad \text{note that } 15i^2 = -15$$

$$= \frac{-3+29i}{34}$$

$$= -\frac{3}{34}+\frac{29}{34}i$$

Division of Complex Numbers

Use of the Conjugate

Response 7(c).

$$z = \frac{i}{1+i}$$

$$= \left(\frac{i}{1+i}\right)\left(\frac{1-i}{1-i}\right) \qquad \text{apply distributive law to numerator}$$

$$= \frac{i-i^2}{1^2+1^2} \qquad \text{note that } -i^2 = -(-1) = +1$$

$$= \frac{1+i}{2} \qquad \text{place the real part first}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Division of Complex Numbers

Use of the Conjugate

Response 7(d).

$$z = \frac{2}{i}$$

$$= \left(\frac{2}{i}\right)\left(\frac{-i}{-i}\right)$$ rationalize the denominator

$$= \frac{-2i}{-i^2}$$ note that $-i^2 = -(-1) = 1$

$$= -2i$$

Division of Complex Numbers

~~Use of the Conjugate~~

Response 7(d). Here is an alternate approach …

$$\begin{aligned} z &= \frac{2}{i} \\ &= \frac{2}{i} \times \frac{i}{i} \qquad \text{rationalize the denom. w/o the conjugate} \\ &= \frac{2i}{i^2} \\ &= -2i \end{aligned}$$

So, when a complex number is purely imaginary, then its rationalization process is simply: $(bi) \times i = bi^2 = -b \in \mathbb{R}$

Complex Number Division

Question 8 Simplify the following division processes using the *complex conjugate to rationalize the denominators*:

$$(a)\qquad z=\frac{50}{4-3i}$$

$$(b)\qquad z=\frac{2+\pi i}{3+7i}$$

$$(c)\qquad z=\frac{i}{1+\sqrt{3}i}$$

$$(d)\qquad z=\frac{8}{5i}$$

Tutorial 4
Complex Numbers

Complex Conjugate, z*

Representation of ℂ-numbers

Question 6. State and read the conjugate, z^*, of the following complex numbers.

$(a)\quad z=-2-4i$

$(b)\quad z=12+15i$

$(c)\quad z=3-8i$

$(d)\quad z=54\angle 90^\circ$

$(e)\quad z=11\angle 27.1^\circ$

$(f)\quad z=\sqrt{5}\angle 4^c$

$(g)\quad z=2i\sin 6^\circ$

$(h)\quad z=\cos 75^\circ - i\sin 75^\circ$

$(i)\quad z=2\left(\cos 15^\circ + i\sin 15^\circ\right)$

$(j)\quad z=5e^{4i}$

$(k)\quad z=25e^{\frac{\pi i}{8}}$

$(l)\quad z=100e^{i}$

No calculations necessary!

Tutorial Solutions
Complex Numbers

Question 6 State the conjugate of the following complex numbers

$(a)\quad z = -2 - 4i$

$(b)\quad z = 12 + 15i$

$(c)\quad z = 3 - 8i$

$(d)\quad z = 54\angle 90^\circ$

$(e)\quad z = 11\angle 27.1^\circ$

$(f)\quad z = \sqrt{5}\angle 4^c$

Solution 6 For each, we <u>negated</u> the imaginary part or the argument.

$(a)\quad z^* = -2 + 4i$

$(b)\quad z^* = 12 - 15i$

$(c)\quad z^* = 3 + 8i$

$(d)\quad z^* = 54\angle -90^\circ$

$(e)\quad z^* = 11\angle -27.1^\circ$

$(f)\quad z^* = \sqrt{5}\angle -4^c$

Tutorial Solutions
Complex Numbers

Question 6 State the conjugate of the following complex numbers

$$(g) \qquad z = 2i\sin 6^\circ$$

$$(h) \qquad z = \cos 75^\circ - i\sin 75^\circ$$

$$(i) \qquad z = 2\left(\cos 15^\circ + i\sin 15^\circ\right)$$

$$(j) \qquad z = 5e^{4i}$$

$$(k) \qquad z = 25e^{\frac{\pi i}{8}}$$

$$(l) \qquad z = 100e^{i}$$

Solution 6 For each, we <u>negated</u> the imaginary part or the argument.

$$(g) \qquad z^* = 2i\sin(-6^\circ)$$

$$(h) \qquad z^* = \cos 75^\circ + i\sin 75^\circ$$

$$(i) \qquad z^* = 2\left(\cos 15^\circ - i\sin 15^\circ\right)$$

$$(j) \qquad z^* = 5e^{-4i}$$

$$(k) \qquad z^* = 25e^{-\frac{\pi i}{8}}$$

$$(l) \qquad z^* = 100e^{-i}$$

Complex Conjugate, z*

Representation of ℂ-numbers

Question 7.

For each of the following, compute the value of zz^*.

$(a) \quad z = 1 + i$

$(b) \quad z = i$

$(c) \quad z = 3 + 4i$

$(d) \quad z = 7\angle 15^\circ$

$(e) \quad z = 2\left(\cos 60^\circ + i \sin 60^\circ\right)$

$(f) \quad z = 9e^{2i}$

Tutorial Solutions
Complex Numbers

Solution 7(a).

Given $z = 1 + i$

then $zz^* = (1+i)(1-i)$

$= 1^2 + 1^2$

$= 2$

Solution 7(b).

Given $z = i$

then $zz^* = i \times (-i)$

$= 1$

NB: Rectangular Rationalization: $(a+ib)(a-ib) = a^2 + b^2$

Tutorial Solutions
Complex Numbers

Solution 7(c).

Given $z = 3 + 4i$

then $$zz^* = (3+4i)(3-4i)$$
$$= 3^2 + 4^2$$
$$= 25$$

Solution 7(d).

Given $z = 7\angle 15^\circ$

then $$zz^* = \left(7\angle 15^\circ\right)\left(7\angle -15^\circ\right)$$
$$= 49$$

NB: Polar Rationalization: $(r\angle\theta)(r\angle -\theta) = r^2$

Tutorial Solutions
Complex Numbers

Solution 7(e).

Given the trig form

$$z = 2\left(\cos 60^\circ + i\sin 60^\circ\right),$$

then
$$\begin{aligned} zz^* &= |z|^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

Solution 7(f).

Given the exponential form

$$z = 9e^{2i},$$

then
$$\begin{aligned} zz^* &= |z|^2 \\ &= 9^2 \\ &= 81 \end{aligned}$$

Reader: I will redo these 2 solutions on the next pages.

Tutorial Solutions
Complex Numbers

Proof 7(e). The trig form: $z = |z|(\cos\theta + i\sin\theta)$

Given $z = 2(\cos 60^\circ + i\sin 60^\circ)$, then

$$\begin{aligned} zz^* &= \left[2(\cos 60^\circ + i\sin 60^\circ)\right] \times \left[2(\cos 60^\circ - i\sin 60^\circ)\right] \\ &= 2^2(\cos^2 60^\circ + \sin^2 60^\circ) \\ &= 4 \times 1 \\ &= 4 \\ &\equiv |z|^2 \end{aligned}$$

Note the Sum of Square rationalization results!

Tutorial Solutions
Complex Numbers

Proof 7(f). The exponetial form: $z = |z| e^{i\theta}$

Given $z = 9e^{2i}$, then

$$
\begin{aligned}
zz^* &= \left(9e^{2i}\right) \times \left(9e^{-2i}\right) \\
&= 9^2 e^0 \\
&= 81 \times 1 \\
&= 81 \\
&\equiv |z|^2
\end{aligned}
$$

So, again, for all 4 forms, the following results hold true:

$$
\begin{aligned}
zz^* &= a^2 + b^2 \\
&= |z|^2
\end{aligned}
$$

Complex Number Division

Question 8 Simplify the following division processes using the *complex conjugate to rationalize the denominators*:

$$(a)\qquad z=\frac{50}{4-3i}$$

$$(b)\qquad z=\frac{2+\pi i}{3+7i}$$

$$(c)\qquad z=\frac{i}{1+\sqrt{3}i}$$

$$(d)\qquad z=\frac{8}{5i}$$

Tutorial Solutions
Complex Numbers

Solution 8(a).

$$z = \frac{50}{4-3i}$$

$$= \left(\frac{50}{4-3i}\right)\left(\frac{4+3i}{4+3i}\right)$$

do not expand numerator in haste

$$= \frac{50(4+3i)}{4^2+3^2}$$

$$= \frac{50(4+3i)}{25}$$

$$= 8+6i$$

Rationalization:

$$(a+ib)(a-ib) = a^2+b^2$$

Tutorial Solutions
Complex Numbers

Solution 8(b).

$$z = \frac{2+\pi i}{3+7i}$$

$$= \left(\frac{2+\pi i}{3+7i}\right)\left(\frac{3-7i}{3-7i}\right)$$ apply distributive law to numerator

$$= \frac{6-14i+3\pi i-7\pi i^2}{3^2+7^2}$$ note: $-7\pi i^2 = 7\pi$ since $i^2 = -1$

$$= \frac{6+7\pi+(3\pi-14)i}{58}$$

$$= \frac{6+7\pi}{58} + \frac{3\pi-14}{58}i$$

Tutorial Solutions
Complex Numbers

Solution 8(c).

$$z = \frac{i}{1+\sqrt{3}i}$$

$$= \left(\frac{i}{1+\sqrt{3}i}\right)\left(\frac{1-\sqrt{3}i}{1-\sqrt{3}i}\right)$$

apply distributive law to numerator

$$= \frac{i-\sqrt{3}i^2}{1^2+\left(\sqrt{3}\right)^2}$$

but $i^2 = -1$

$$= \frac{\sqrt{3}}{4} + \frac{1}{4}i$$

place the real part at the front

Tutorial Solutions
Complex Numbers

Solution 8(d).

$$
\begin{aligned}
z &= \frac{8}{5i} \\
&= \left(\frac{8}{5i}\right)\left(\frac{-5i}{-5i}\right) && \text{rationalize the denominator w/ conjugate} \\
&= \frac{-40i}{25} && \text{rationalisation result is always positive} \\
&= -\frac{8}{5}i
\end{aligned}
$$

Tutorial Solutions
Complex Numbers

Solution 8(d). Here is an alternate approach …

$$z = \frac{8}{5i}$$

$$= \frac{8}{5i} \times \frac{i}{i} \qquad \text{rationalize the denom. w/o the conjugate}$$

$$= -\frac{8i}{5}$$

So, when a complex number is pure imaginary, then its rationalization process is simply: $(bi) \times i = bi^2 = -b \in \mathbb{R}$

Tutorial Solutions
Complex Numbers

Revelation 8(d). Finally, we arrive as a reusable fact!

The reciprocal of i using rectangular and polar form:

(a)
$$\frac{1}{i} = \frac{1}{i} \times \left(\frac{i}{i}\right)$$
$$= \frac{i}{i^2}$$
$$= -i$$

(b)
$$\frac{1}{i} = \frac{1}{1\angle 90^\circ}$$
$$= 1\angle -90^\circ$$
$$= -i$$

Hence, it is worth remembering that, for all numbers, "to divide by i is to multiply by $-i$." That is, $\frac{z}{i} = -iz$

Complex Numbers

Thank You

For more information, please visit:
http://www.boszik.net/mse