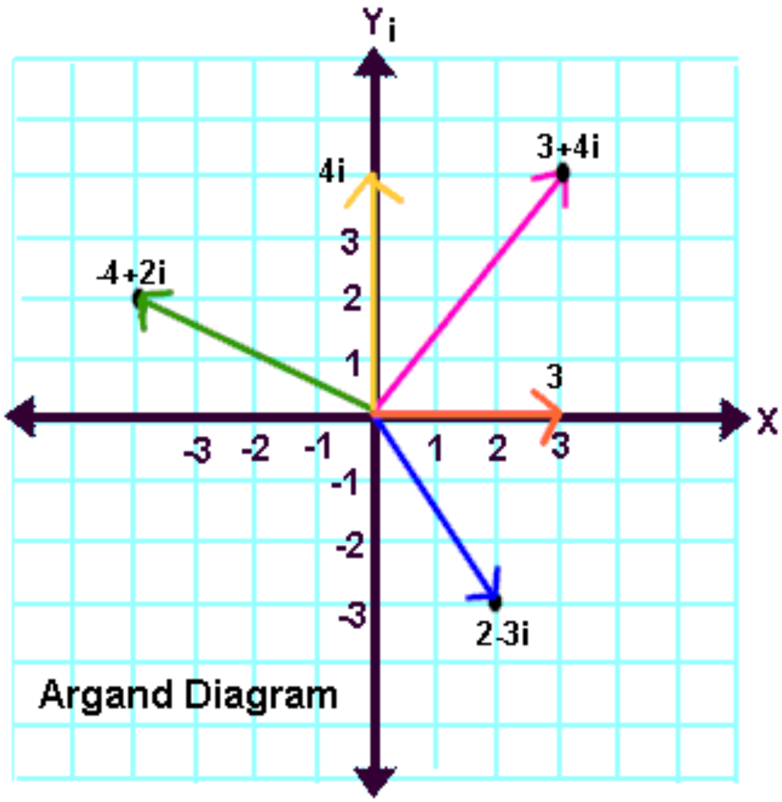
Complex Numbers



TUTORIAL 2

1. Definition and the Argand Diagram
2. Decomposition of the Rectangular Form
3. Illustrative Examples

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The 21st Century

Definition of the Complex Number, z

Recall the definition of the unit '*imaginary number*':

$$\exists i: \quad i^2 = -1 \quad \Leftrightarrow \quad i = \sqrt{-1}$$

Then the following are now possible:

$$\begin{aligned} z_1 &= \sqrt{-16} \\ &= \sqrt{16 \times (-1)} \\ &= \sqrt{16} \times \sqrt{-1} \\ &= 4 \times i \\ &= 4i \end{aligned}$$

$$\begin{aligned} z_2 &= \sqrt{-80} \\ &= \sqrt{80 \times (-1)} \\ &= \sqrt{16 \times 5} \times \sqrt{-1} \\ &= 4\sqrt{5} \times i \\ &= 4\sqrt{5}i \end{aligned}$$

Definition of the Complex Number, z

Example 1. Solve the following equations.

$$(a) \qquad x = \sqrt{-17}$$

$$(b) \qquad x^2 + 100 = 0$$

$$(c) \qquad x^2 + x = -1$$

Response 1(a)

$$\begin{aligned} x &= \sqrt{-17} \\ &= \sqrt{17} \times \sqrt{-1} \\ &= \sqrt{17} i \end{aligned}$$

Definition of the Complex Number, z

Response 1(b)

$$x^2 + 100 = 0$$

then $x^2 = 100$

such that $x = \pm\sqrt{-100}$ note the "$\pm$"

$= \pm\sqrt{100} \times \sqrt{-1}$ this step can be skipped

$= \pm 10i$

Definition of the Complex Number, z

Response 1(c) The given equation is quadratic.

$$x^2 + x = -1 \quad \Leftrightarrow \quad x^2 + x + 1 = 0 \qquad \text{(not easily factorised!)}$$

So, using the quadratic formula, we get

$$x = \frac{-1 \pm \sqrt{1^2 - (4 \times 1 \times 1)}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

The 2 quadratic roots are complex as the discriminant is negative!

Definition of the Complex Number, z

Question 1. Solve the following equations for z.

(a) $z = \sqrt{-400}$

(b) $z^2 + 36 = 0$

(c) $2 + z^2 = 0$

Question 2. Solve the following equations for z.

(a) $z^2 - 2z + 3 = 0$

(b) $z^2 + 36 = 0$

(c) $z^2 - 4z + 5 = 0$

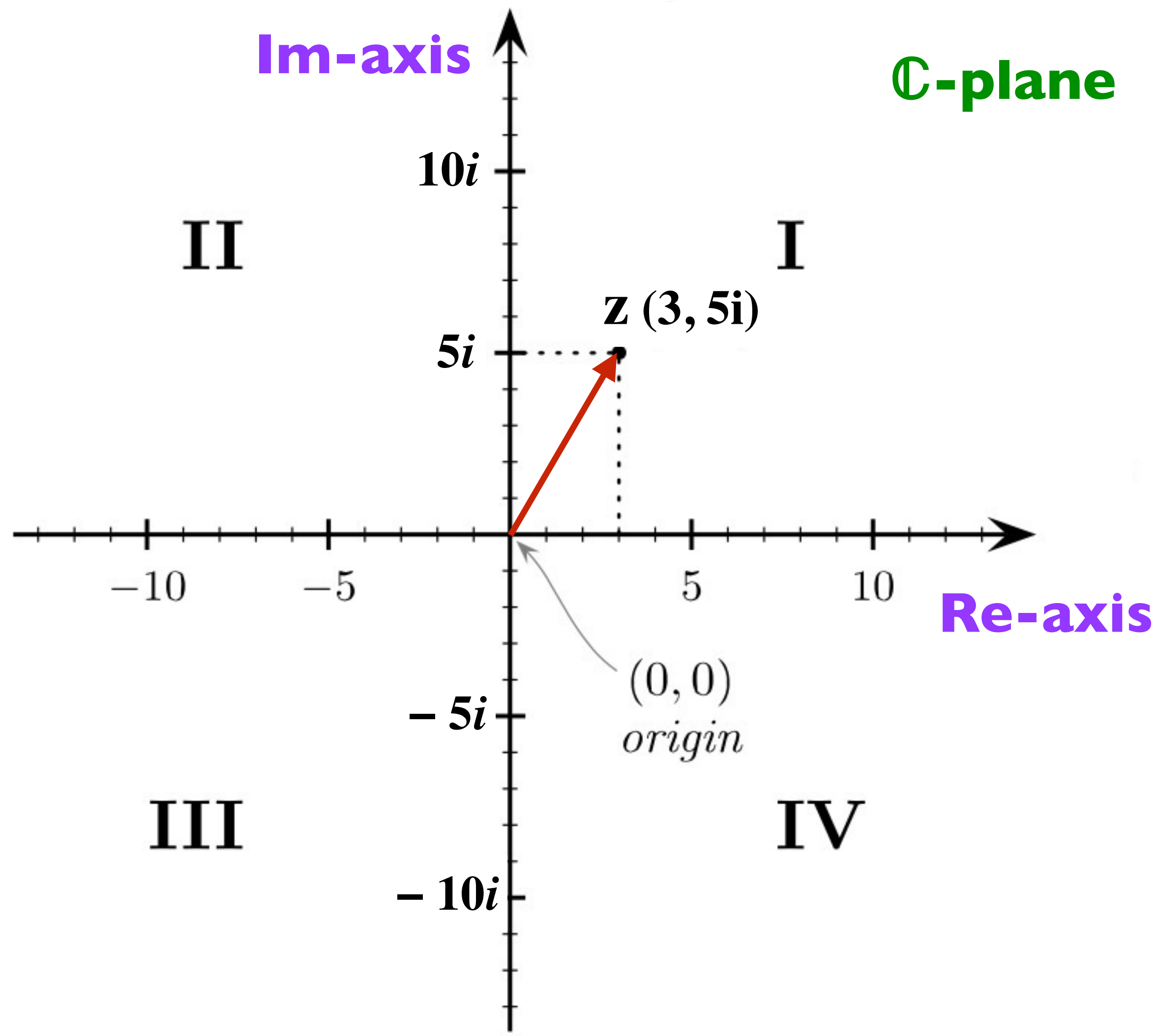
The Argand Diagram & ℂ-Number Decompositions

Consider a complex number in rectangular form: $z = 3 + 5i$

It has …

- real part, a
- imaginary part, b
- length, r
- angle, θ

Let's practice finding each.



The Argand Diagram & $\mathbb{C}$-Number Decompositions

Example 2. *Sketch* each of the following complex number on an Argand diagram. Then, find the *real part*, *imaginary part*, *modulus* and *argument* of each.

$$(a) \qquad z_A = 3 + 5i$$

$$(b) \qquad z_B = -10 + 10i$$

$$(c) \qquad z_C = -5 - 9i$$

$$(d) \qquad z_D = 3 - 4i$$

For all arguments, report principal angles in the range:

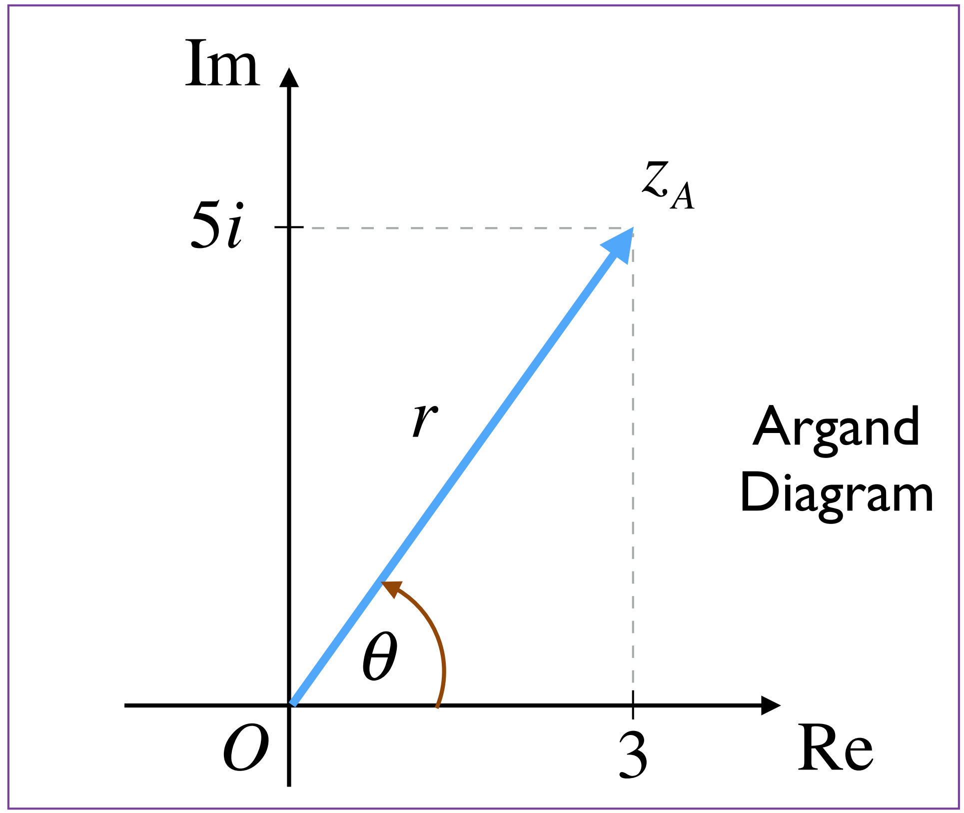
$$-180^\circ < \theta \leq 180^\circ$$

The Argand Diagram & ℂ-Number Decompositions

Response 2(a). A sketch of $z_A = 3+5i$ is shown. Importantly, it lies in Quadrant I.

$$
\begin{aligned}
a &= \operatorname{Re} z_A \\
&= \operatorname{Re}(3+5i) \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
b &= \operatorname{Im} z_A \\
&= \operatorname{Im}(3+5i) \\
&= 5
\end{aligned}
$$

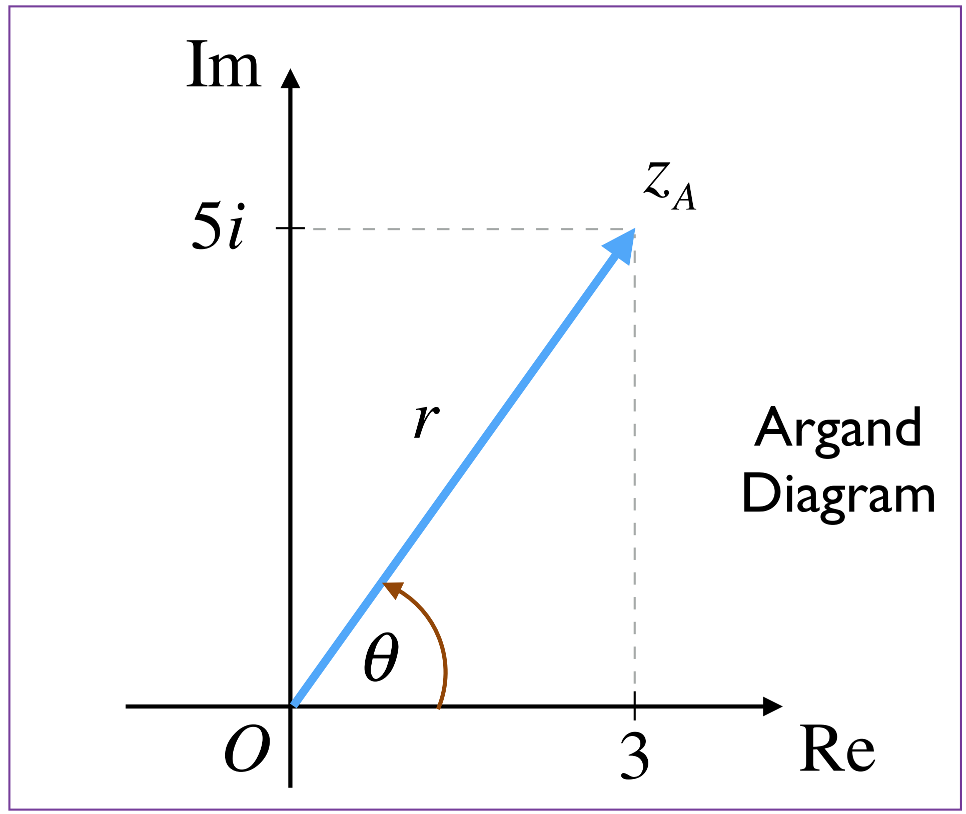


The Argand Diagram & $\mathbb{C}$-Number Decompositions

Response 2(a). A sketch of $z_A = 3+5i$ is shown. Importantly, it lies in Quadrant I.

$$r = |z_A| = \sqrt{3^2+5^2} = \sqrt{34}\ \text{units}$$

$$\theta = \arg z_A = \tan^{-1}\left(\frac{5}{3}\right) = 59.0^\circ$$

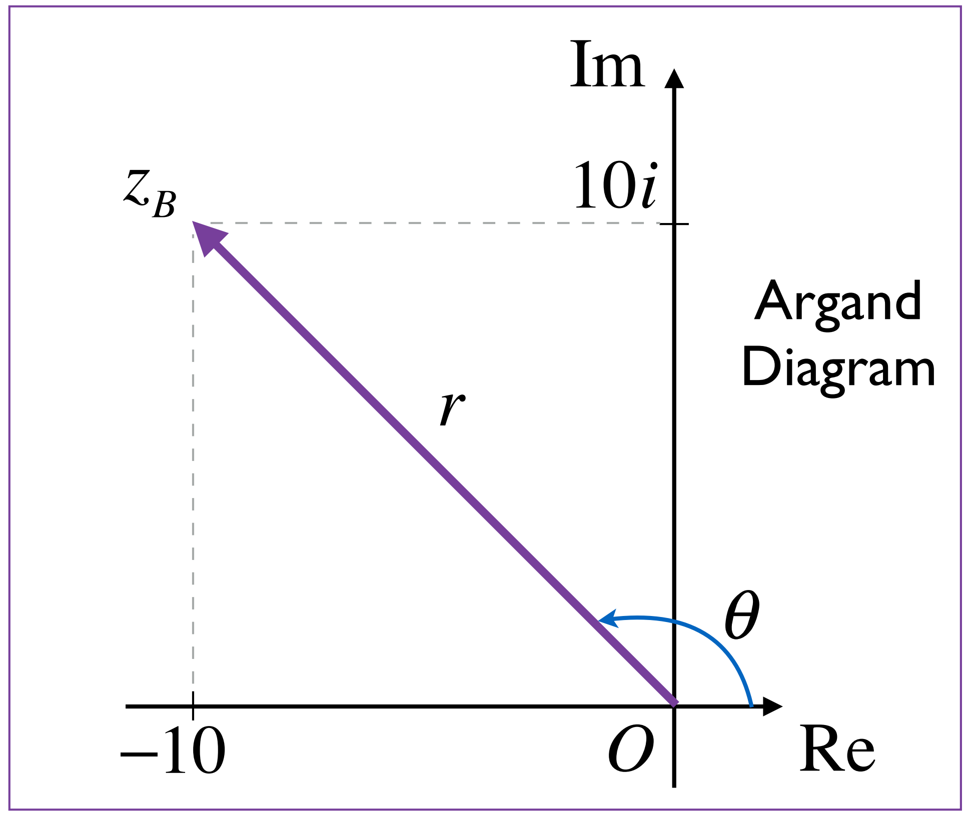


The Argand Diagram & ℂ-Number Decompositions

Response 2(b). A sketch $z_B = -10 + 10i$ is shown. Importantly, it lies in Quadrant II.

$$a = \operatorname{Re} z_B$$
$$= \operatorname{Re}(-10 + 10i)$$
$$= -10$$

$$b = \operatorname{Im} z_B$$
$$= \operatorname{Im}(-10 + 10i)$$
$$= 10$$

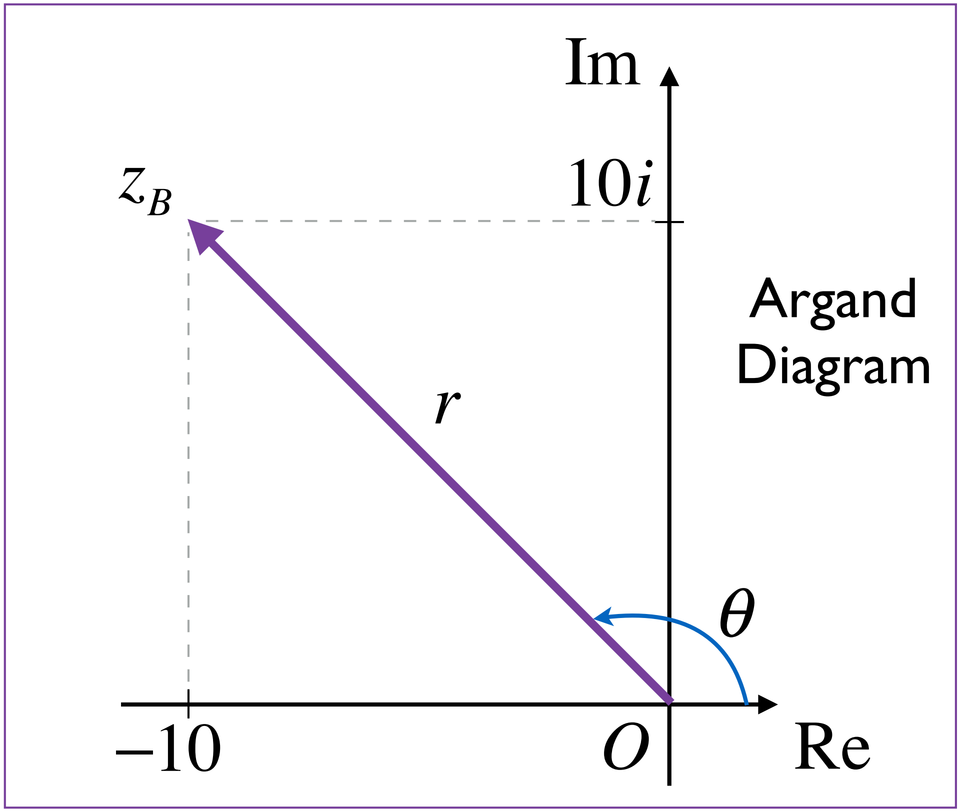


The Argand Diagram & ℂ-Number Decompositions

Response 2(b). A sketch $z_B = -10 + 10i$ is shown. Importantly, it lies in Quadrant II.

$$r = |z_B|$$
$$= \sqrt{10^2 + 10^2}$$
$$= 10\sqrt{2} \text{ units}$$

$$\theta = \arg z_B$$
$$= 180^\circ + \tan^{-1}\left(\frac{10}{-10}\right)$$
$$= 135^\circ$$

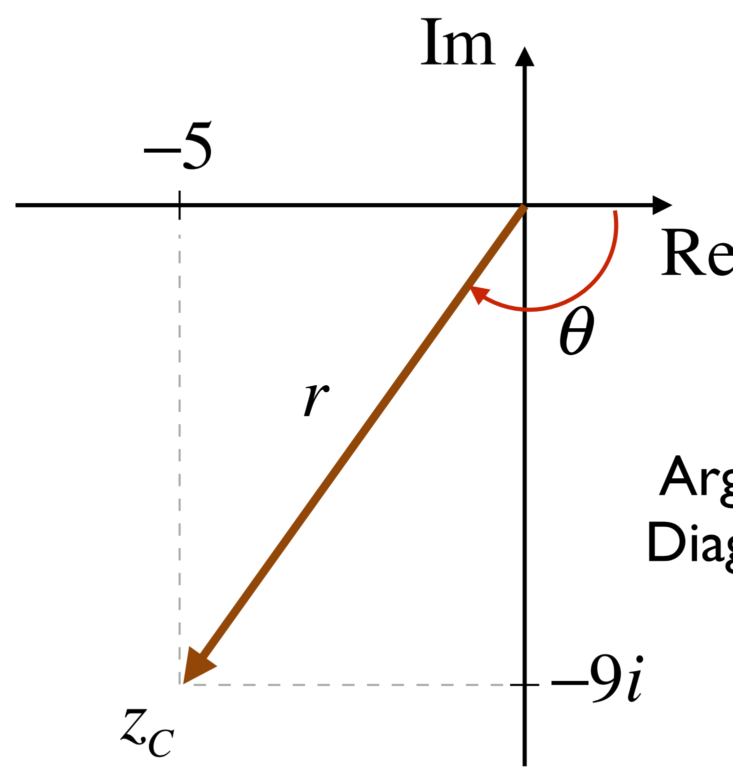


The Argand Diagram & ℂ-Number Decompositions

Response 2(c). A sketch of $z_C = -5 - 9i$ is shown. Importantly, it lies in Quadrant III.

$$a = \operatorname{Re} z_C$$
$$= \operatorname{Re}(-5 - 9)$$
$$= -5$$

$$b = \operatorname{Im} z_C$$
$$= \operatorname{Im}(-5 - 9)$$
$$= -9$$



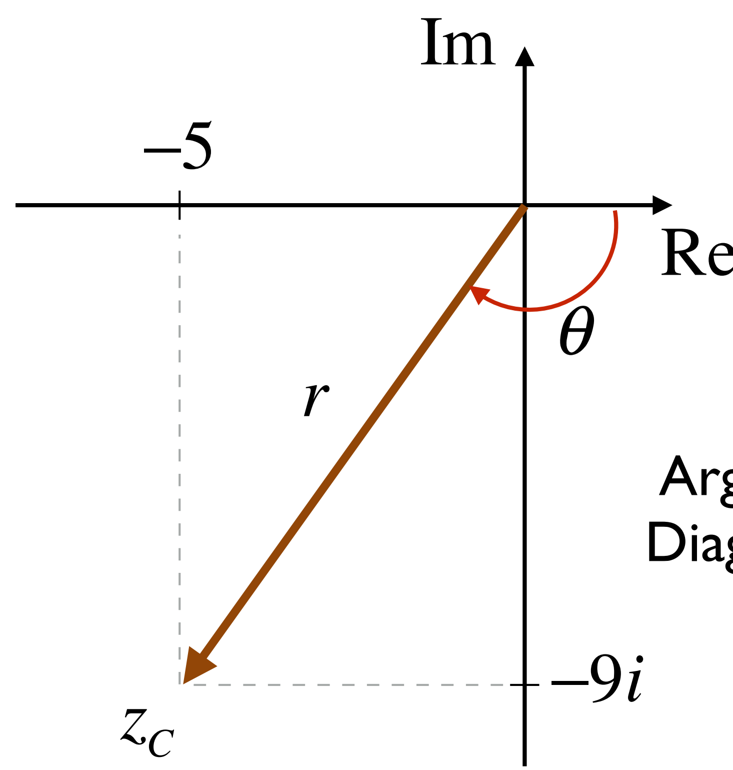
Argand Diagram

The Argand Diagram & ℂ-Number Decompositions

Response 2(c). A sketch of $z_C = -5 - 9i$ is shown. Importantly, it lies in Quadrant III.

$$r = |z_C|$$
$$= \sqrt{5^2 + 9^2}$$
$$= \sqrt{106} \text{ units}$$

$$\theta = \arg z_C$$
$$= -180^\circ + \tan^{-1}\left(\frac{-9}{-5}\right)$$
$$= -119.1^\circ$$



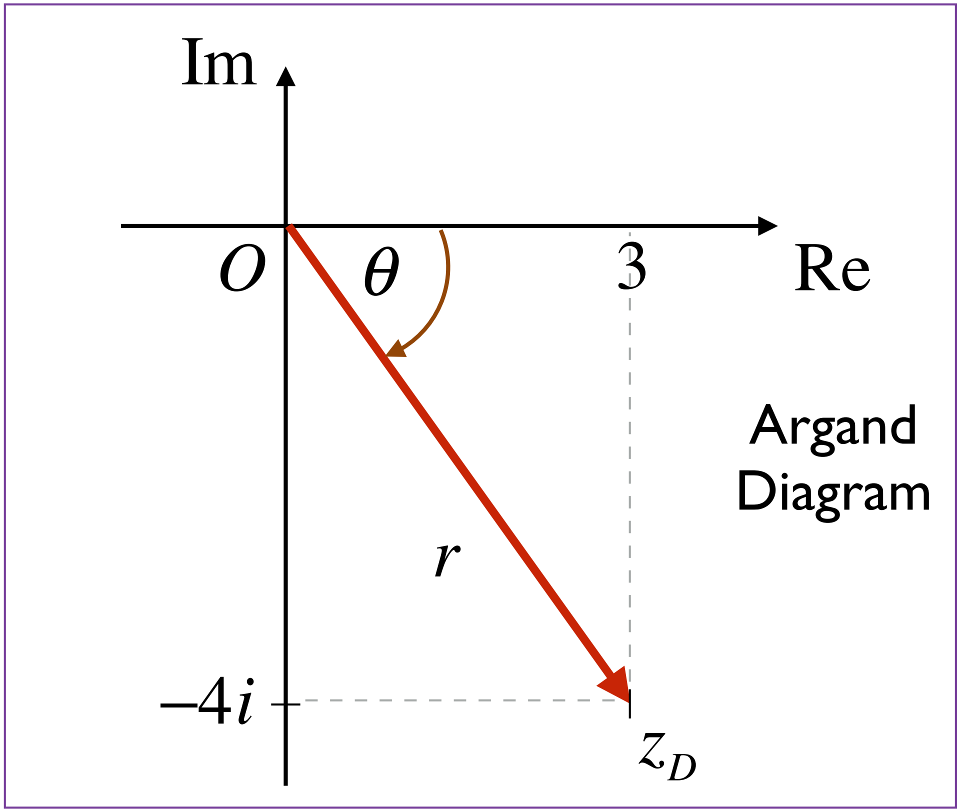
Argand Diagram

The Argand Diagram & $\mathbb{C}$-Number Decompositions

Response 2(d). A sketch of $z_D = 3 - 4i$ is shown. Importantly, it lies in Quadrant IV.

$$a = \operatorname{Re} z_D$$
$$= \operatorname{Re}(3 - 4i)$$
$$= 3$$

$$b = \operatorname{Im} z_D$$
$$= \operatorname{Im}(3 - 4i)$$
$$= -4$$

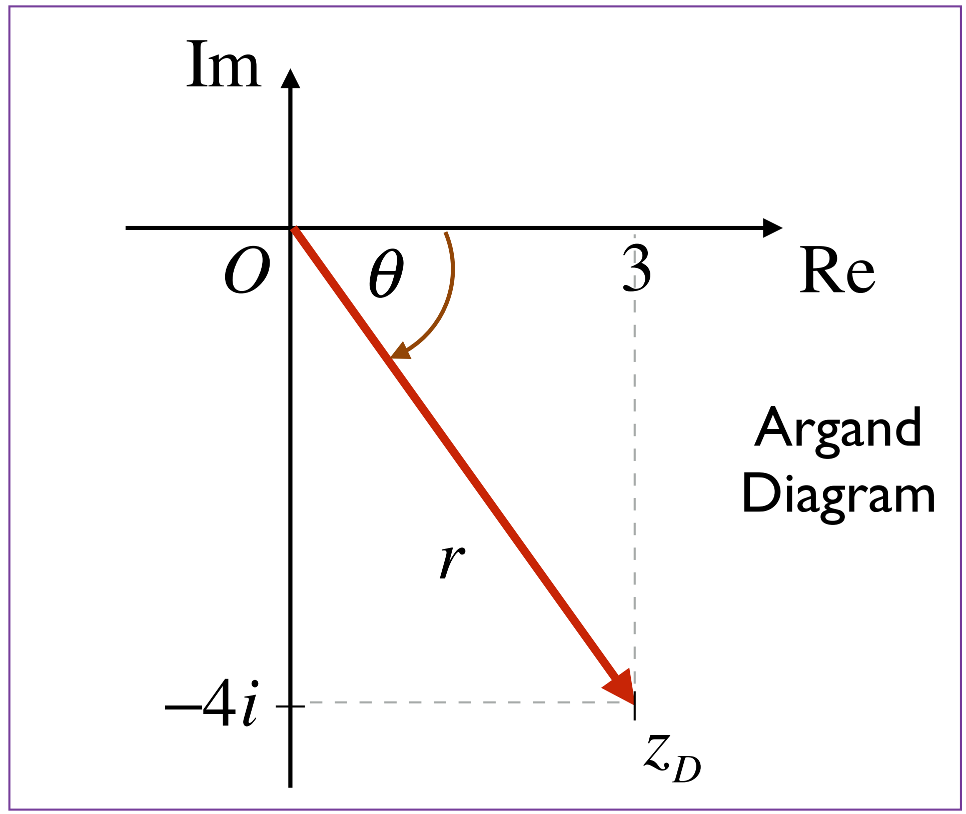


The Argand Diagram & $\mathbb{C}$-Number Decompositions

Response 2(d). A sketch of $z_D = 3 - 4i$ is shown. Importantly, it lies in Quadrant IV.

$$r = |z_D| = \sqrt{3^2 + 4^2} = 5 \text{ units}$$

$$\theta = \arg z_D = \tan^{-1}\left(\frac{-4}{3}\right) = -36.9^\circ$$



Complex Number Decompositions

Question 3. *Sketch* each of the following complex number on an Argand diagram. Then, find the *real part, imaginary part, modulus* and *argument* of each.

$$(a) \qquad z_1 = 3 + 4i$$

$$(b) \qquad z_2 = -1 + i$$

$$(c) \qquad z_3 = -\sqrt{3} - 2i$$

$$(d) \qquad z_4 = 5 - 7i$$

For all arguments, report principal angles in the range:

$$-180^\circ < \theta \leq 180^\circ$$

Tutorial 2
Complex Numbers

Tutorial Solutions
Complex Numbers

Question 1. Solve the following equations for z.

$(a) \qquad z = \sqrt{-400}$

$(b) \qquad z^2 + 36 = 0$

$(c) \qquad 2 + z^2 = 0$

Solution 1(a)

$$z = \sqrt{-400}$$

$$= \sqrt{400} \times \sqrt{-1}$$ <--- skippable step

$$= 20i$$

Tutorial Solutions
Complex Numbers

Solution 1(b).

$$z^2 + 36 = 0$$
$$z^2 = -36$$
$$z = \pm\sqrt{-36}$$
$$z = \pm 6i$$

Solution 1(c).

$$2 + z^2 = 0$$
$$z^2 = -2$$
$$z = \pm\sqrt{-2}$$
$$z = \pm\sqrt{2}i$$

Comments: In these solutions, the technique used is transposing and taking the square root of both sides. The results are *conjugate pairs*.

Tutorial Solutions
Complex Numbers

Question 2. Solve the following equations for z.

$$(a) \qquad z^2 - 2z + 3 = 0$$

$$(b) \qquad z^2 + 36 = 0$$

$$(c) \qquad z^2 - 4z + 5 = 0$$

Solution 2.

These 3 quadratic problems give rise to *complex quadratic roots* since the *discriminant of each of the equations is negative.*

Tutorial Solutions
Complex Numbers

Solution 2(a).

Given $z^2 - 2z + 3 = 0$, then $b^2 - 4ac = 4 - 12$

$$= -8 > 0 \quad \Rightarrow \quad z \in \mathbb{C}$$

So, using the *quadratic formula*, the 2 complex roots are:

$$z = \frac{-(-2) \pm \sqrt{-8}}{2 \times 1}$$

$$= \frac{2 \pm 2\sqrt{2}i}{2}$$

$$= 1 \pm \sqrt{2}i$$

Tutorial Solutions
Complex Numbers

Solution 2(b).

Given $z^2 + 36 = 0$, then $b^2 - 4ac = 0 - (4)(36) > 0 \quad \Rightarrow \quad z \in \mathbb{C}$

Since $b = 0$, using the *quadratic formula* is less desirable.

Let's factorize and using the *difference of 2 squares*, for fun! So,

$$\begin{aligned} z^2 + 36 &= 0 \\ z^2 - (-36) &= 0 \\ z^2 - (6i)^2 &= 0 \\ (z+6i)(z-6i) &= 0 \quad \Rightarrow \quad z = \pm 6i \end{aligned}$$

Note $\sqrt{-36} = 6i$

$\Leftrightarrow -36 = (6i)^2$

Tutorial Solutions
Complex Numbers

Solution 2(c).

Given $z^2 - 4z + 5 = 0$, then $b^2 - 4ac = 16 - 20$

$$= -4 > 0 \quad \Rightarrow \quad z \in \mathbb{C}$$

So, using the *quadratic formula*, the 2 complex roots are:

$$z = \frac{-(-4) \pm \sqrt{-4}}{2 \times 1}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

Tutorial Solutions
Complex Numbers

Question 3. *Sketch* each of the following complex number on an Argand diagram. Then, find the *real part*, *imaginary part*, *modulus* and *argument* of each.

$$(a) \qquad z_1 = 3 + 4i$$

$$(b) \qquad z_2 = -1 + i$$

$$(c) \qquad z_3 = -\sqrt{3} - 2i$$

$$(d) \qquad z_4 = 5 - 7i$$

For all arguments, report principal angles in the range:

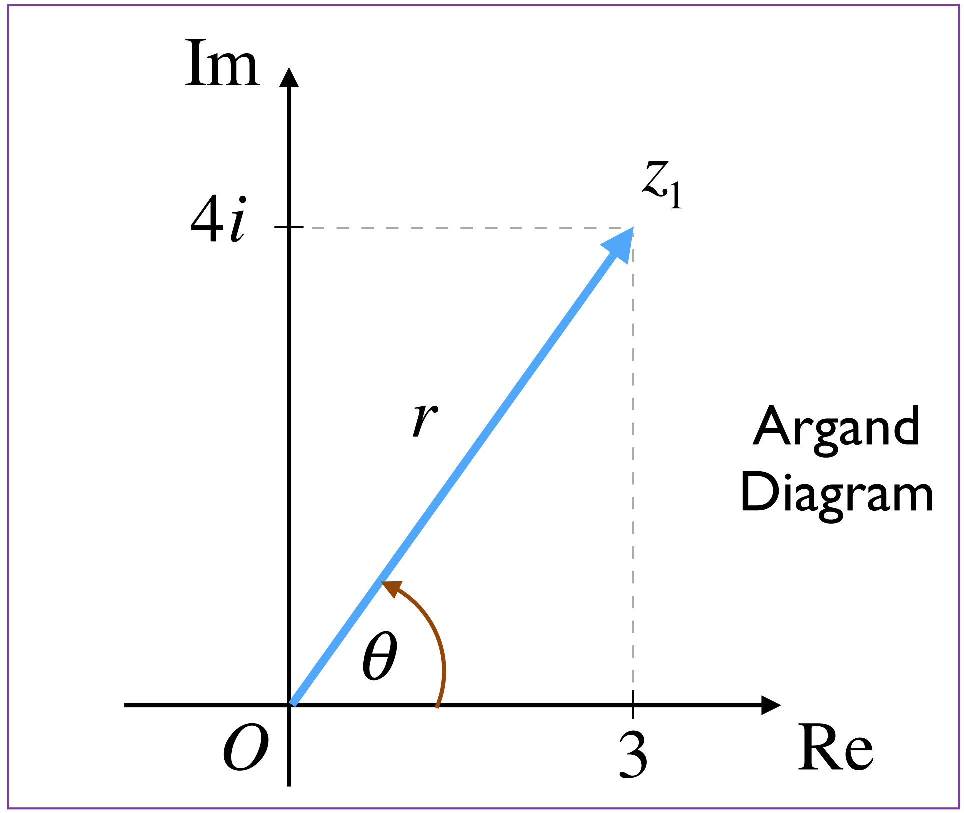
$$-180^\circ < \theta \le 180^\circ$$

Tutorial Solutions
Complex Numbers

Solution 3(a). The sketch of $z_1 = 3+4i$ is shown. Importantly, it lies in Quadrant I.

$$
\begin{aligned}
a &= \operatorname{Re} z_1 \\
&= \operatorname{Re}(3+4i) \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
b &= \operatorname{Im} z_1 \\
&= \operatorname{Im}(3+4i) \\
&= 4
\end{aligned}
$$

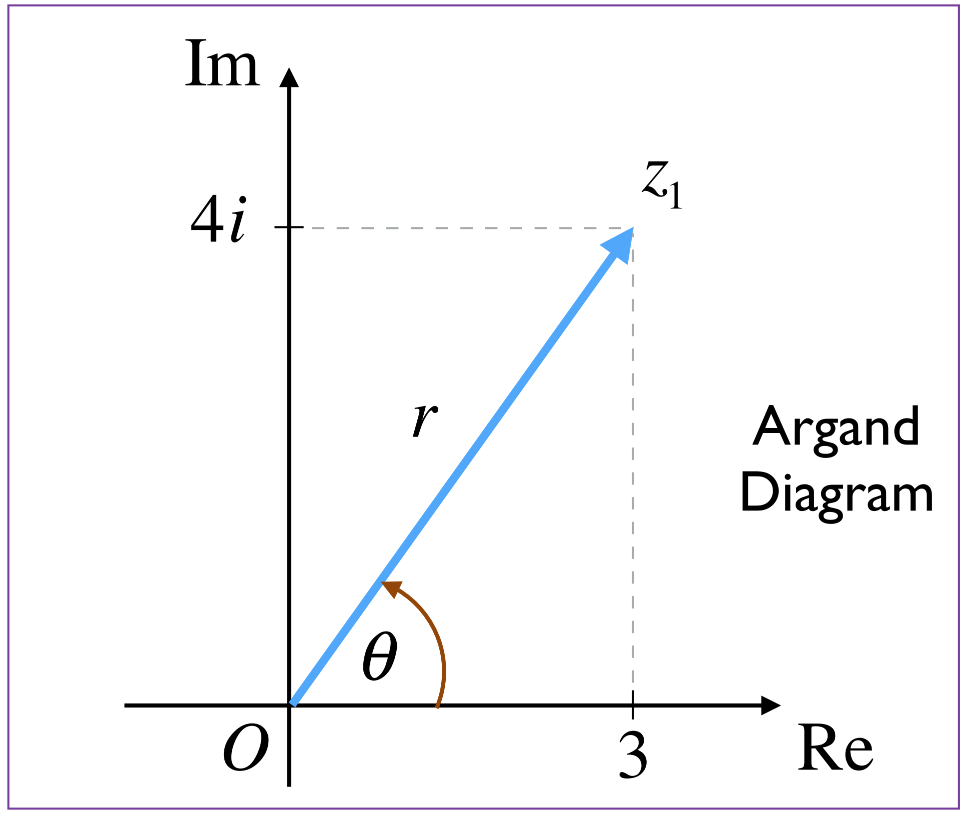


Tutorial Solutions
Complex Numbers

Solution 3(a). The sketch of $z_1 = 3 + 4i$ is shown. Importantly, it lies in Quadrant I.

$$r = |z_1| = \sqrt{3^2 + 4^2} = 5 \text{ units}$$

$$\theta = \arg z_1 = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$$

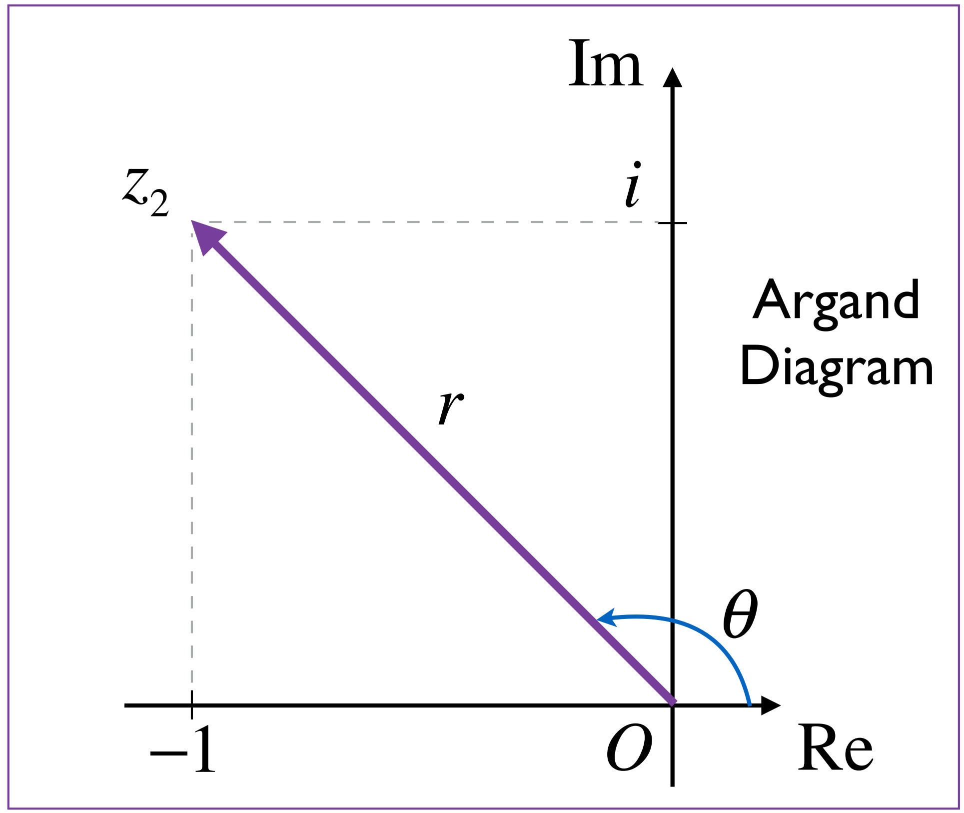


Tutorial Solutions
Complex Numbers

Solution 3(b). The sketch of $z_2 = -1 + i$ is shown. Importantly, it lies in Quadrant II.

$$a = \operatorname{Re} z_2$$
$$= \operatorname{Re}(-1 + i)$$
$$= -1$$

$$b = \operatorname{Im} z_2$$
$$= \operatorname{Im}(-1 + i)$$
$$= 1$$

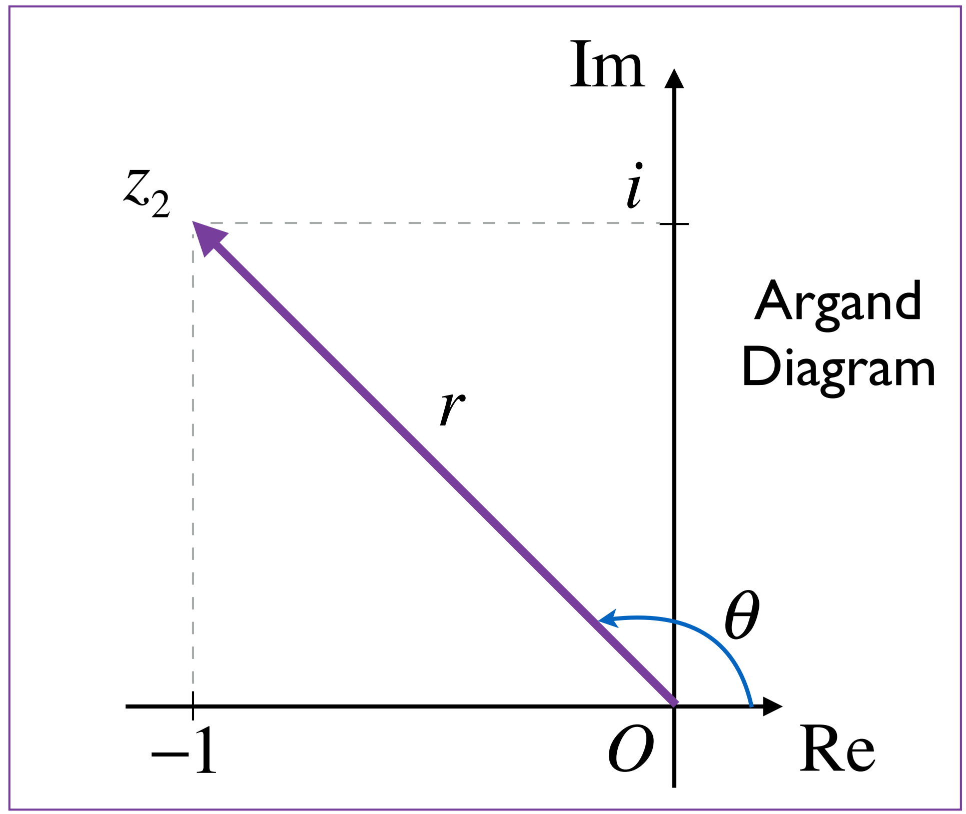


Tutorial Solutions
Complex Numbers

Solution 3(b). The sketch of $z_2 = -1 + i$ is shown. Importantly, it lies in Quadrant II.

$$r = |z_2|$$
$$= \sqrt{1^2 + 1^2}$$
$$= \sqrt{2} \text{ units}$$

$$\theta = \arg z_2$$
$$= 180^\circ + \tan^{-1}\left(\frac{1}{-1}\right)$$
$$= 135^\circ$$

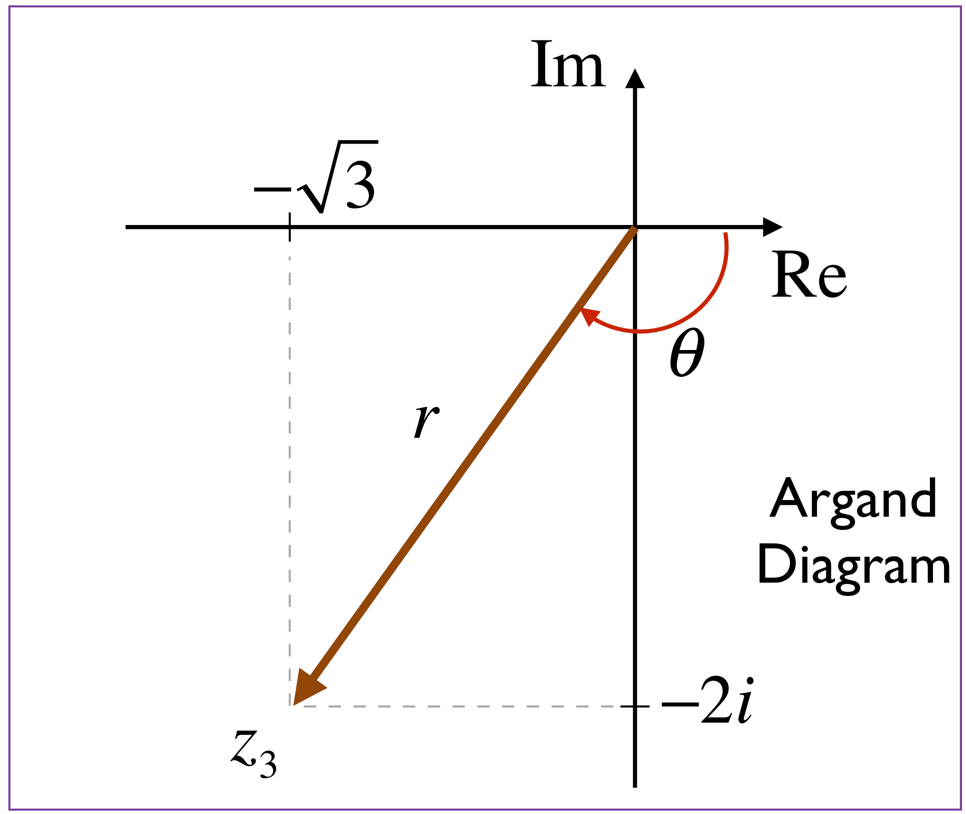


Tutorial Solutions
Complex Numbers

Solution 3(c). The sketch of $z_3 = -\sqrt{3} - 2i$ is shown. Importantly, it lies in Quadrant III.

$$a = \operatorname{Re} z_3$$
$$= \operatorname{Re}\left(-\sqrt{3} - 2i\right)$$
$$= -\sqrt{3}$$

$$b = \operatorname{Im} z_3$$
$$= \operatorname{Im}\left(-\sqrt{3} - 2i\right)$$
$$= -2$$

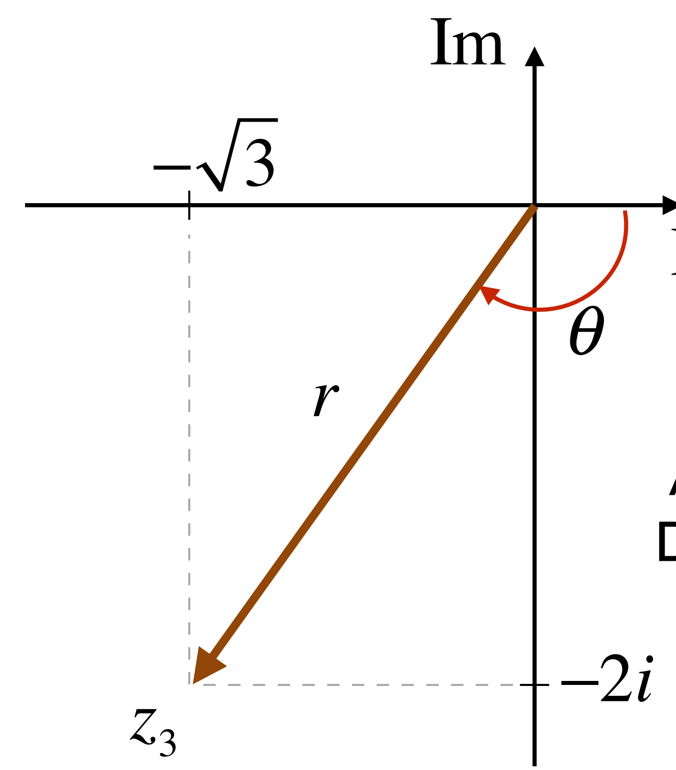


Tutorial Solutions
Complex Numbers

Solution 3(c). The sketch of $z_3 = -\sqrt{3} - 2i$ is shown. Importantly, it lies in Quadrant III.

$$r = |z_3|$$
$$= \sqrt{(\sqrt{3})^2 + 2^2}$$
$$= \sqrt{7} \text{ units}$$

$$\theta = \arg z_3$$
$$= -180^\circ + \tan^{-1}\left(\frac{-2}{-\sqrt{3}}\right)$$
$$= -130.9^\circ$$



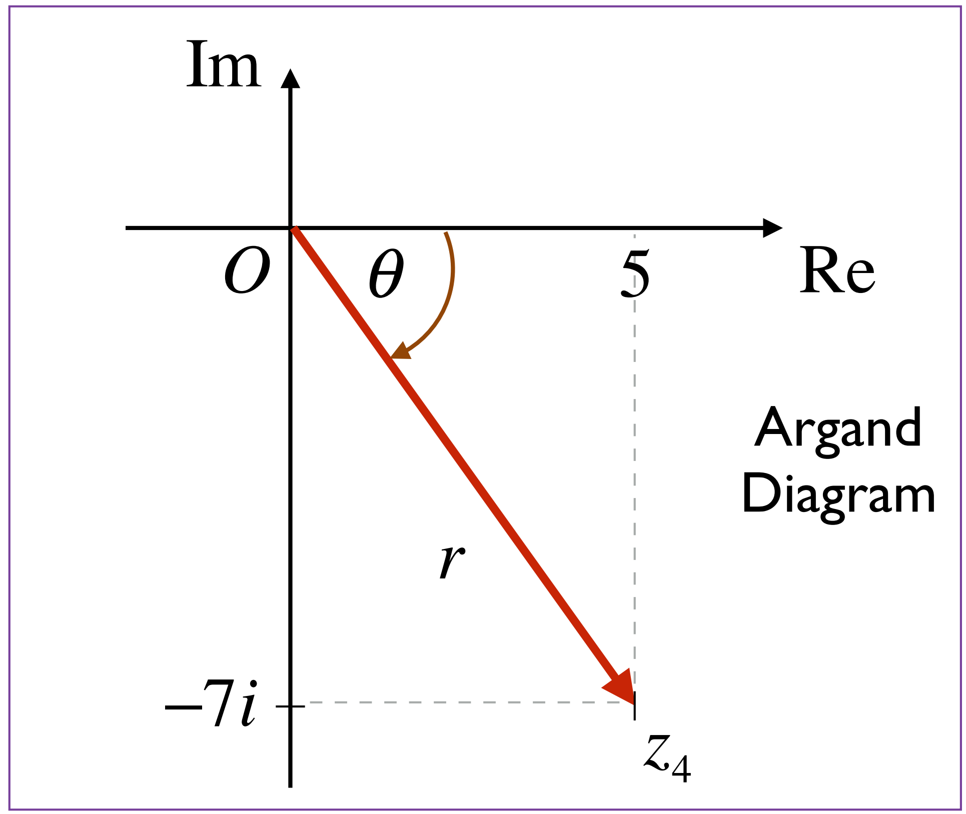
Argand Diagram

Tutorial Solutions
Complex Numbers

Solution 3(d). The sketch of $z_4 = 5 - 7i$ is shown. Importantly, it lies in Quadrant IV.

$$a = \mathrm{Re}\, z_4 = \mathrm{Re}(5 - 7i) = 5$$

$$b = \mathrm{Im}\, z_4 = \mathrm{Im}(5 - 7i) = -7$$

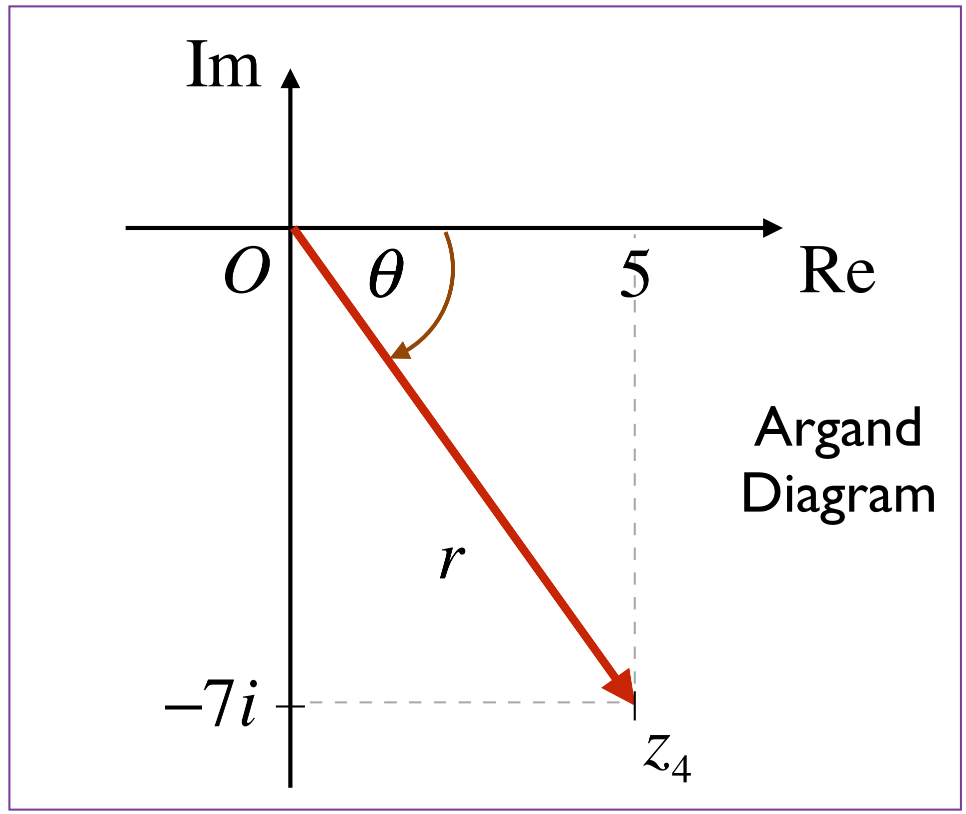


Tutorial Solutions
Complex Numbers

Solution 3(d). The sketch of $z_4 = 5 - 7i$ is shown.
Importantly, it lies in Quadrant IV.

$$r = |z_4|$$
$$= \sqrt{5^2 + 7^2}$$
$$= \sqrt{74} \text{ units}$$

$$\theta = \arg z_4$$
$$= \tan^{-1}\left(\frac{-7}{5}\right)$$
$$= -54.5^\circ$$



Complex Numbers

Thank You

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